Supplemental Notes

EE503 Week 03 Dr. Franzke

HW #3

1. HW #3 handout
2. Leon Garcia 2.44 – 2.47
   2.58 – 2.61

Topics

- Higher dimension probability spaces
- Counting and binomial theorem
- Limits and infinite series
- Probability limits
- Borel-Cantelli Lemma.

Defn: Cartesian product $A \times B$

$$A \times B = \{(x,y) : x \in A \text{ AND } y \in B\}$$

Ex:

- $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$
- 3 coin flips:

$$\longleftrightarrow \quad \Omega^3 = \Omega \times \Omega \times \Omega$$

$$\{H,T\} \times \{H,T\} \times \{H,T\}.$$

Defn: Borel sigma algebra on $\mathbb{R}^n \longrightarrow \mathcal{B}(\mathbb{R}^n)$

$$\mathcal{B}(\mathbb{R}^n) = \sigma(\{\text{interval products}\})$$

$$= \sigma(\{(-\infty, a_1] \times \cdots (-\infty, a_n] : a_k \in \mathbb{R} \text{ if } k = 1, \ldots, n\})$$

Defn: $n! = n(n-1) \cdots 3 \cdot 2 \cdot 1$ for $n \in \{0, 1, 2, \ldots\}$.

$(0! = 1)$

Fact: (Stirling's Approximation)

$$\ln n! \approx n \cdot \ln n - n$$

$$\approxeq n \cdot \ln n$$

Defn: Gamma function $\Gamma(\alpha)$ for $\alpha > 0$.

$$\Gamma(\alpha) = \int_0^\infty \underset{\text{power function}}{x^{\alpha-1}} \underbrace{e^{-x}}_{\text{exponential function}} dx$$

$$\therefore \Gamma(\alpha+1) = \alpha \cdot \Gamma(\alpha) \quad \text{if } \alpha \in \mathbb{R}^+$$

☆ $$\therefore \Gamma(n+1) = n! \quad \text{if } n \in \mathbb{Z}^+.$$

Defn: Permutation = Self-Bijection

- $f : S \to S$
- f : 1-to-1 (injection)
- f : onto (surjection)

Fundamental Rule of Accounting

arrangement = | n_1 | n_2 | ... | n_k |

$\therefore$ $|\text{arrangements}| = \prod_{j=1}^{k} n_j$

$(= n^k$ if $n_j = n \ \forall j)$

$P(n,k)$ = # permutations of length k on n elements

$= \dfrac{n!}{(n-k)!}$ for $k \leq n$.

$C(n,k)$ = # combinations of length k on n elements

$= \dbinom{n}{k} = \dfrac{n!}{(n-k)!\,k!}$ for $k \leq n$.

Thrm: (Pascal's Formula) $\dbinom{n}{k} = \dbinom{n-1}{k-1} + \dbinom{n-1}{k}$

$\therefore$ Pascal's Triangle

```
0            1
1          1   1
2        1   2   1
3      1   3   3   1
4    1   4   6   4   1
5  1   5  10  10   5   1
⋮            ⋮
```

$\binom{4}{2}$

$(x+y)^0 = 1$

$(x+y)^1 = x+y$

$(x+y)^2 = x^2 + 2xy + y^2$

$(x+y)^3$

$(x+y)^4$

$\vdots$

Review: how to change the INDEX of summation

Show: $\sum_{k=1}^{n} a_{k-1} = \sum_{j=0}^{n-1} a_j$

Technique: $\sum_{k=1}^{n} a_{k-1} = \sum_{k=1}^{k=n} a_{k-1}$ put $j = k-1$ $\therefore k = j+1$

$$= \sum_{j+1=1}^{j+1=n} a_j$$

$$= \sum_{j=0}^{j=n-1} a_j$$

$$= \sum_{j=0}^{n-1} a_j$$

QED.

Thrm: (Binomial Theorem)

$$(p+q)^n = \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k}.$$

Prf: (by induction on $n = 1, 2, 3, \ldots$)

Basis: $n = 1$ $\therefore (p+q)^1 = p+q.$

$$\sum_{k=0}^{1} \binom{1}{k} p^k q^{n-k} = \binom{1}{0} p^0 q^1 + \binom{1}{1} p^1 q^0$$

$$= p+q.$$

QED (basis)

Induction Step Induction hypothesis: $(p+q)^{n-1} = \sum_{k=0}^{n-1} \binom{n-1}{k} p^k q^{(n-1)-k}$

$$\therefore (p+q)^n = (p+q)(p+q)^{n-1}$$

$$\overset{IH}{=} (p+q) \sum_{k=0}^{n-1} \binom{n-1}{k} p^k q^{(n-1)-k}$$

$$= \sum_{k=0}^{n-1} \binom{n-1}{k} p^{k+1} q^{n-k-1} + \sum_{k=0}^{n-1} \binom{n-1}{k} p^k q^{n-k}$$

$$= \underbrace{\sum_{j=1}^{n} \binom{n-1}{j-1} p^j q^{n-j}}_{j=k+1 \quad \therefore k=j-1} + \underbrace{\sum_{j=0}^{n-1} \binom{n-1}{j} p^j q^{n-j}}_{j=k}$$

$$= \left[\underbrace{\binom{n-1}{n-1}}_{=1} \underbrace{p^n q^{n-n}}_{p^n} + \sum_{j=1}^{n-1} \binom{n-1}{j-1} p^j q^{n-j} \right] +$$

$$\left[\sum_{j=1}^{n-1} \binom{n-1}{j} p^j q^{n-j} + \underbrace{\binom{n-1}{0}}_{=1} \underbrace{p^0 q^{n-0}}_{q^n} \right]$$

$$= p^n + \sum_{j=1}^{n-1} \binom{n-1}{j-1} p^j q^{n-j} + \sum_{j=1}^{n-1} \binom{n-1}{j} p^j q^{n-j} + q^n.$$

$$= p^n + \sum_{j=1}^{n-1} \left[\underbrace{\binom{n-1}{j-1} + \binom{n-1}{j}}_{=\binom{n}{j}} \right] p^j q^{n-j} + q^n$$

$$= p^n + \sum_{j=1}^{n-1} \binom{n}{j} p^j q^{n-j} + q^n$$

$$= \binom{n}{n} p^n q^{n-n} + \sum_{j=1}^{n-1} \binom{n}{j} p^j q^{n-j} + \binom{n}{0} p^0 q^{n-0}$$

$$= \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k}$$

QED.

Binomial pdf

(probability "mass" density function)

$$b(n,k,p) = \binom{n}{k} p^k (1-p)^{n-k} \quad \text{if} \quad k \le n \quad \text{and} \quad p \in [0,1].$$

p = "success probability"

= P[Success on one Bernoulli Trial].

$\therefore$ $b(n,k,p) = P[\,k \text{ successes in } n\text{-trials}\,]$.

$$P[X \le k] = P[X=0] + P[X=1] + \cdots + P[X=k] = \sum_{j=0}^{k} b(n,j,p)$$

$\therefore$ $\sum_{k=0}^{n} b(n,k,p) = 1$ by binomial theorem.

since: $(p + (1-p))^n = 1^n = 1$.

Multinomial Theorem:

$$(p_1 + \cdots + p_k)^n = \sum_{l_1+\cdots+l_k=n} \frac{n!}{l_1!\, l_2! \cdots l_k!} p_1^{l_1} \cdots p_k^{l_k}$$

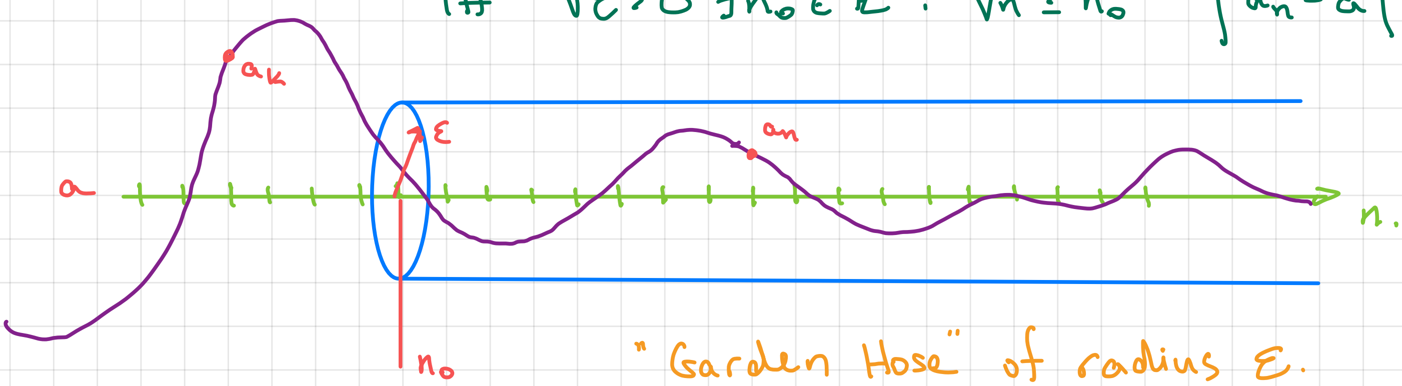
(partition)

if $l_1 + \cdots + l_k = n$

Defn: Limit of a sequence $a_1, a_2, a_3, \ldots$

$a_n \longrightarrow a$ iff $\lim_{n\to\infty} a_n = a$

iff $\forall \varepsilon > 0\ \exists n_0 \in \mathbb{Z}^+ : \forall n \geq n_0 \quad |a_n - a| < \varepsilon$



"Garden Hose" of radius ε.

$\therefore$ smaller $\varepsilon > 0 \longrightarrow$ Takes longer to find n_0.

Partial Sum $\quad S_N = \sum_{n=0}^{N} a_n = a_0 + a_1 + \cdots + a_N$

Infinite Series $\quad$ " $\sum_{n=0}^{\infty} a_n = S$ " converges

iff $\lim_{N\to\infty} S_N = S$ $\quad$ (S finite)

iff $\forall \varepsilon > 0\ \exists n_0 \in \mathbb{Z}^+ : \forall n \geq n_0 : \ |S_N - S| < \varepsilon$.

(else "diverges")

Special case:

$$\lim_{n\to\infty} a_n = \infty \quad \text{iff} \quad \forall \varepsilon > 0\ \exists n_0 \in \mathbb{Z}^+ : \forall n \geq n_0 : a_n > \varepsilon.$$

- a type of non-convergence
- not the same in general as $\lim_{n\to\infty} a_n \neq a$.

Infinite Products

Thm: Suppose $a_n > 0$ $\forall n$ $(a_n \in \mathbb{R}^+)$ then

$$\prod_{n=1}^{\infty} (\underbrace{1+a_n}_{\to 1}) \text{ converges iff } \sum_{n=1}^{\infty} \underbrace{a_n}_{\to 0} < \infty$$

Ex: $\prod_{n=1}^{\infty} \left(1+\frac{1}{n^p}\right)$ converges iff $p > 1$.

Corr: Suppose $0 < p_n < 1$ (as when p_n is a probability).

$$\therefore \underbrace{\prod_{n=1}^{\infty} (1-p_n) > 0}_{\text{converges}} \quad \text{iff} \quad \sum_{n=1}^{\infty} p_n < \infty$$

Fact: $\lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n = e^x$ where $e^x \triangleq \sum_{n=0}^{\infty} \frac{x^n}{n!}$ (proof later)

Fact: $\sum_{n=0}^{\infty} a_n = s \xrightarrow[(\not\leftarrow)]{} \lim_{n\to\infty} a_n = 0$

Test: p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$

- converges if $p > 1$.
- diverges if $p \leq 1$

Test: (Alternating Series test)

$\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges if
- $a_k \geq a_{k+1} > 0 \ \forall k \in \mathbb{Z}^+$.
- $\lim_{n \to \infty} a_n = 0$

Defn: $\sum_{n=0}^{\infty} a_n$ converges absolutely iff $\sum_{n=0}^{\infty} |a_n|$ converges

(∴ treat infinite sums like finite sums)

Facts: (1) Absolute convergence $\longrightarrow$ convergence ($\not\longleftarrow$)

(2) If
- $\sum_{n=1}^{\infty} a_n$ converges absolutely
- $\sum_{n=1}^{\infty} b_n$ is any rearrangement of $\sum_{n=1}^{\infty} a_n$

Then:
- $\sum_{n=1}^{\infty} b_n$ converges
- $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n$

Defn: $\sum_{n=0}^{\infty} a_n$ converges conditionally iff
- $\sum_{n=0}^{\infty} a_n$ converges
- $\sum_{n=0}^{\infty} |a_n|$ diverges

Thm: (Riemann's Rearrangement Theorem) Pick ANY $x \in \mathbb{R}$. Suppose $\sum_{n=0}^{\infty} a_n$ converges conditionally. Then $\exists$ a re-arrangement $\sum_{n=0}^{\infty} b_n$ of $\sum_{n=0}^{\infty} a_n$ where $\sum_{n=0}^{\infty} b_n = x$.

(example: $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$.

☆

Ratio test for convergence of $\sum_{n=0}^{\infty} a_n$ (series)

Put $L \stackrel{\Delta}{=} \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$ Then

(1) $L < 1 \implies$ series converges absolutely.

(2) $L > 1 \implies$ series diverges

(3) $L = 1 \implies$ test fails

Ex: Geometric Series: $a_n = a^n$. $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} a^n$ $(a \neq 0)$

$$\therefore \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{a^{n+1}}{a^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{\cancel{a^n} \cdot a}{\cancel{a^n}} \right|$$

$$= \lim_{n\to\infty} |a|.$$

$$= |a|.$$

$\therefore \sum_{n=0}^{\infty} a^n$ converges absolutely if $|a| < 1$.

Fact: $|x| \leq c$ iff $-c \leq x \leq c$ $(c > 0)$

Power Series for variable x

$$\sum_{n=0}^{\infty} a_n \cdot x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

Ex: Find all x so that $\sum_{n=0}^{\infty} \frac{n}{5^n} x^n$ converges absolutely.

By the ratio test:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)x^{n+1}}{n \cdot x^n} \cdot \frac{5^n}{5^{n+1}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(n+1)x}{5n} \right|$$

$$= \underbrace{\lim_{n\to\infty} \left(\frac{n+1}{n}\right)}_{=1} \cdot \frac{1}{5} \cdot |x|.$$

$$= \frac{1}{5}|x|$$

$$< 1$$

iff $|x| < 5$.

iff $-5 < x < 5$

diverges if $|x| > 5$

diverges at $x = 5$ & $x = -5$.

Thrm: (1) If $\sum_{n=0}^{\infty} a_n x^n$ converges for $c \neq 0$ then it converges absolutely $|x| < |c|$.

(2) If $\sum_{n=0}^{\infty} a_n x^n$ diverges for $d \neq 0$ then it diverges if $|x| > |d|$.

Thrm: Exactly one holds for $\sum_{n=0}^{\infty} a_n x^n$

(1) it converges only if $x = 0$

(2) it converges absolutely for all x (e.g. e^x)

(3) $\exists r > 0$: $\begin{cases} \text{It converges absolutely if } |x| < r. \\ \quad \text{and} \\ \text{It diverges if } |x| > r. \end{cases}$

$\therefore$ Case 3: "radius of convergence" $r > 0$

$(-r, r)$

Important Result:

Thrm: $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x \quad \forall x \in \mathbb{R}.$

Prf:

Case 1: $x = 0$

$$\therefore \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n \Big|_{x=0} = \lim_{n\to\infty} (1)^n = 1 = e^0$$

Case 2: $x \neq 0$

$$\ln\left(\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n\right) = \lim_{n\to\infty} \ln\left(1+\frac{x}{n}\right)^n$$

since $\ln x$ is continuous. and continuity implies sequential continuity.

$$= \lim_{n\to\infty} n \cdot \ln\left(1+\frac{x}{n}\right)$$

$$= \frac{x}{x} \cdot \lim_{n\to\infty} n \cdot \ln\left(1+\frac{x}{n}\right) \qquad x \neq 0.$$

$$= x \cdot \lim_{n\to\infty} \frac{n}{x} \cdot \ln\left(1+\frac{x}{n}\right)$$

$$= x \cdot \lim_{n \to \infty} \frac{\ln(1+x/n)}{x/n}$$

$$= x \cdot \lim_{n \to \infty} \frac{\ln\left(1+\frac{x}{n}\right) - 0}{x/n}$$

$$= x \cdot \lim_{n \to \infty} \frac{\ln\left(1+\frac{x}{n}\right) - \ln 1}{x/n}.$$

$$= x \cdot \left(\lim_{h \to 0} \frac{\ln(1+h) - \ln 1}{h} \right)$$

if $h = \frac{x}{n}$ (since x is fixed)

$$= x \cdot \left(\lim_{h \to 0} \frac{\ln(w+h) - \ln(w)}{h} \bigg|_{w=1} \right)$$

$$= x \cdot \left(\frac{d}{dw} \ln w \bigg|_{w=1} \right)$$

defn derivative

$$= x \cdot \left(\frac{1}{w} \bigg|_{w=1} \right)$$

$$= x$$

$\therefore \lim_{n \to \infty} \left(1+\frac{x}{n}\right)^n = e^x$ by exponentiation.

QED

Probability Limits.

Define: (1) $\bigcup_{n=1}^{\infty} A_n = \lim_{n\to\infty} A_n$ if $A_k \subset A_{k+1}$ $\forall k \in \mathbb{N}$

"increasing sequence"

(2) $\bigcap_{n=1}^{\infty} A_n = \lim_{n\to\infty} A_n$ if $A_k \supset A_{k+1}$ $\forall k \in \mathbb{N}$

"decreasing sequence"

Overview: Continuity of Probability $\longrightarrow$ Boole's Inequality $\longrightarrow$ Borel Cantelli

☆ **Thrm:** (Continuity of Probability)

Given $(\Omega, \mathcal{A}, P)$

(1) $P(\lim_{n\to\infty} A_n) = \lim_{n\to\infty} P(A_n) = \lim_{n\to\infty} P\left(\bigcup_{k=1}^{n} A_k\right)$

if $A_k \subset A_{k+1}$ $\forall k \in \mathbb{N}$.

(2) $P(\lim_{n\to\infty} A_n) = \lim_{n\to\infty} P(A_n) = \lim_{n\to\infty} P\left(\bigcap_{k=1}^{n} A_k\right)$

if $A_k \supset A_{k+1}$ $\forall k \in \mathbb{N}$.

Prf: Define $\{B_n\}$ as:

$$\begin{cases} (1) & B_1 = A_1 \\ (2) & B_n = A_n \cap A_{n-1}^c = A_n - A_{n-1} \quad (\text{if } n>1) \end{cases}$$

Proof uses 3 Lemmas (proofs below)

- **Lemma 1:** $\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n$

- **Lemma 2:** $\{B_n\}$ pairwise disjoint : $B_i \cap B_j = \emptyset$ if $i \neq j$.

- **Lemma 3:** $P(B_n) = P(A_n) - P(A_{n-1})$

Proof of (1):

$$P\left(\lim_{n\to\infty} A_n\right) = P\left(\bigcup_{n=1}^{\infty} A_n\right)$$ defn limit A_n

$$= P\left(\bigcup_{n=1}^{\infty} B_n\right)$$ Lemma 1.

$$= \sum_{n=1}^{\infty} P(B_n)$$ Lemma 2 + CAT.

$$= \lim_{n\to\infty} \sum_{k=1}^{n} P(B_k)$$ defn inf. series.

$$= \lim_{n\to\infty} \left(P(A_1) + \sum_{k=2}^{n} P(B_k) \right)$$ defn B_1

$$= \lim_{n\to\infty} \left(P(A_1) + \sum_{k=2}^{n} \left(P(A_k) - P(A_{k-1})\right)\right)$$ Lemma 3

$$= \lim_{n\to\infty} \left(P(\cancel{A_1}) + (P(\cancel{A_2}) - P(\cancel{A_1})) + (P(\cancel{A_3}) - P(\cancel{A_2})) + \cdots + (P(A_n) - P(\cancel{A_{n-1}})\right)$$

$$= \lim_{n\to\infty} P(A_n)$$

QED (1)

Proof of (2)

Decreasing sequence $A_1 \supset A_2 \supset A_3 \supset \cdots$ but $A_{k+1}^c \supset A_k^c$ $\forall k \in \mathbb{N}$.
(increasing sequence)

by contraposition: $A \subset B$ iff $B^c \subset A^c$.

$$P\left(\lim_{n\to\infty} A_n\right) = P\left(\bigcap_{n=1}^{\infty} A_n\right)$$ defn lim A_n and decreasing.

$$= P\left(\left(\bigcup_{n=1}^{\infty} A_n^c\right)^c\right)$$ DeMorgans

$$= 1 - P\left(\bigcup_{n=1}^{\infty} A_n^c\right)$$ $P(A^c) = 1 - P(A)$

$$= 1 - \lim_{n\to\infty} P(A_n^c)$$ (1) and contraposition

$$= 1 - \lim_{n\to\infty} \left(1 - P(A_n)\right)$$ $P(A^c) = 1 - P(A)$

$= \cancel{x} - \cancel{x} + \lim_{n\to\infty} P(A_n)$ limit of sum

$= \lim_{n\to\infty} P(A_n)$

QED (2)

Proof of Lemmas:

Lemma 1: $\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n$

Prf: Claim 1: $\bigcup_{n=1}^{\infty} B_n \subset \bigcup_{n=1}^{\infty} A_n$

Prf: pick $x \in \bigcup_{n=1}^{\infty} B_n$ Ass.

$\therefore \exists k \in \mathbb{N} : x \in B_k$ defn $\cup$

$\therefore x \in A_k \cap A_{k-1}^c$ (or A_1) defn B_n

$\therefore x \in A_k$ defn $\cap$

$\therefore x \in \bigcup_{n=1}^{\infty} A_n$ $A \subset A \cup B$

$\therefore \bigcup_{n=1}^{\infty} B_n \subset \bigcup_{n=1}^{\infty} A_n$ defn $\subset$

Claim 2: $\bigcup_{n=1}^{\infty} A_n \subset \bigcup_{n=1}^{\infty} B_n$

Prf: pick $x \in \bigcup_{n=1}^{\infty} A_n$ Ass.

$\therefore \exists k \in \mathbb{N} . x \in A_k$ defn $\cup$

$\therefore k = 1$ or $k > 1$

Case 1: $k = 1$

$\therefore A_1 = B_1$ defn B_n

$\therefore x \in B_1 \subset \bigcup_{n=1}^{\infty} B_n$ $A \subset A \cup B$

$\therefore \bigcup_{n=1}^{\infty} A_n \subset \bigcup_{n=1}^{\infty} B_n$ defn $\subset$

Case 2: $k > 1$

$\therefore$ Either $x \in A_{k-1}$ or $x \in A_{k-1}^c$ — excluded middle

- Say $x \in A_{k-1}^c$ — Ass.

  $\therefore x \in A_k \cap A_{k-1}^c$ — defn $\cap$

  $\therefore x \in B_k$ — defn B_n

  $\therefore x \in \bigcup_{n=1}^{\infty} B_n$ — $A \subset A \cup B$

  $\therefore \bigcup_{n=1}^{\infty} A_n \subset \bigcup_{n=1}^{\infty} B_n$ — defn $\subset$

- Else $x \in A_{k-1}$ — Ass.

  $\therefore x \in A_{k-2}$ or $x \in A_{k-2}^c$ — excluded middle

  $\vdots \qquad \vdots$

  until $\exists m \in \mathbb{N} : 2 \leq m < k$ & $x \in A_m \cap A_{m-1}^c = B_m$

  since $A_1 \subset A_2 \subset \cdots$ and defn B_n.

  $\therefore x \in \bigcup_{n=1}^{\infty} B_n$ — $A \subset A \cup B$

  $\therefore \bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n$ — defn $\subset$

$\therefore \bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n$ — claim 1+2, defn =

QED (Lemma 1)

Lemma 2: $\{B_n\}$ pairwise disjoint $\quad B_i \cap B_j = \emptyset$ if $i \neq j$.

Prf: Say $i \neq j$ — Ass.

$\therefore B_i \cap B_j = (A_i \cap A_{i-1}^c) \cap (A_j \cap A_{j-1}^c)$ — defn B_n

$= A_i \cap A_{i-1}^c \cap A_j \cap A_{j-1}^c$ — assoc $\cap$

$= A_i \cap A_{j-1}^c \cap A_j \cap A_{i-1}^c$ — comm. $\cap$

$= (A_i \cap A_{j-1}^c) \cap (A_j \cap A_{i-1}^c)$ assoc $\cap$

Case 1: $i < j$

$\therefore A_i \subset A_{j-1}$ — $A_i \subset A_{i+1}$

$\therefore A_i \cap A_{j-1}^c = \phi$ — $A \subset B$ iff $A \cap B^c = \phi$.

$\therefore B_i \cap B_j = (A_i \cap A_{j-1}^c) \cap (A_j \cap A_{i-1}^c)$

$= \phi \cap (A_j \cap A_{i-1}^c)$

$= \phi$

Case 2: $j < i$

$\therefore B_i \cap B_j = (A_i \cap A_{j-1}^c) \cap \phi$ — as in Case 1.

$= \phi$

$\therefore B_i \cap B_j = \phi$ if $i \neq j$

QED (Lemma 2)

Lemma 3: $P(B_n) = P(A_n) - P(A_{n-1})$

Prf: $A_n^c \subset A_{n-1}^c$ — $A_k \subset A_{k+1} \subset \cdots$ and contraposition

$X = A_n \cup A_n^c$ — excluded middle

$\subset A_n \cup A_{n-1}^c$ — above

$\subset X$ — always

$\therefore A_n \cup A_{n-1}^c = X$ — defn $=$

$\therefore P(A_n \cup A_{n-1}^c) = 1$ — CAI

$\therefore P(B_n) = P(A_n \cap A_{n-1}^c)$ — defn B_n

$= P(A_n) + P(A_{n-1}^c) - P(A_n \cup A_{n-1}^c)$ — add. theorem

$$= P(A_n) + P(A_{n-1}^c) - 1 \qquad \text{above}$$

$$= P(A_n) + \cancel{1} - P(A_{n-1}) - \cancel{1} \qquad P(A^c) = 1 - P(A)$$

$$= P(A_n) - P(A_{n-1})$$

QED (lemma 3)

$\therefore$ QED continuity of probability.

Recall: mathematical induction proves the <u>finite</u> Boole's inequality for any n: $P\left(\bigcup_{k=1}^{n} A_k\right) \le \sum_{k=1}^{n} P(A_k)$

General case follows from the Continuity Theorem

<u>Thrm</u>: (Boole's inequality) $P\left(\bigcup_{n=1}^{\infty} A_n\right) \le \sum_{n=1}^{\infty} P(A_n)$

<u>Prf</u>: Define $B_n = \bigcup_{k=1}^{n} A_k$ $\quad \therefore B_1 = A_1$ and $B_1 \subset B_2 \subset B_3 \subset \cdots$

$\therefore \bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n$

$$\therefore P\left(\bigcup_{n=1}^{\infty} A_n\right) = P\left(\bigcup_{n=1}^{\infty} B_n\right) \qquad \text{defn } B_n$$

$$= P\left(\lim_{n\to\infty} B_n\right) \qquad \text{defn } \lim_{n\to\infty} B_n$$

$$= \lim_{n\to\infty} P(B_n) \qquad \text{continuity theorem}$$

$$= \lim_{n\to\infty} P(B_{n-1} \cup A_n) \qquad \text{defn } B_n$$

$$\le \lim_{n\to\infty} \left(P(B_{n-1}) + P(A_n)\right) \qquad \text{additive theorem}$$

$$= \lim_{n\to\infty} \left(P\left(\bigcup_{k=1}^{n-1} A_k\right) + P(A_n)\right) \qquad \text{defn } B_n$$

$$\le \lim_{n\to\infty} \left(\sum_{k=1}^{n-1} P(A_k) + P(A_n)\right) \qquad \text{finite Boole's ineq.}$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} P(A_k)$$

$$= \sum_{n=1}^{\infty} P(A_n)$$ defn partial sum.

Defn: An event A_k occurs infinitely often ("i.o.")

$\longleftrightarrow$ $\forall n \geq 1 \; \exists k \geq n : A_k$ occurs

$\longleftrightarrow$ $\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$

is a "tail event" with probability 0 or 1.

Borel-Cantelli Lemma (Theorem) connect an i.o. event to the convergence or divergence of an infinite series.

Thrm: (Borel-Cantelli Lemma):

(1) if $\sum_{n=1}^{\infty} P(A_n) < \infty$ then $P\left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k\right) = 0$

(2) if $\sum_{n=1}^{\infty} P(A_n) = \infty$ and $\{A_k\}$ are independent then

$$P\left(\bigcap_{n=1}^{\infty} \bigcup_{k\geq n}^{\infty} A_k\right) = 1$$

Prf (1):

Suppose $\sum_{n=1}^{\infty} P(A_n) < \infty$

$\therefore \lim_{n\to\infty} P(A_n) = 0$ since converges

Define $B_n = \bigcup_{k=n}^{\infty} A_k$

$\therefore$ Decreasing $B_k \supset B_{k+1}$

$$\therefore P\left(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} A_k\right) = P\left(\bigcap_{n=1}^{\infty} B_n\right)$$ defn B_n

$$= P\left(\lim_{n\to\infty} B_n\right)$$ defn $\lim_{n\to\infty} A_n$

$$= \lim_{n\to\infty} P(B_n)$$ continuity thrm since $B_k \supseteq B_{k+1}$

$$= \lim_{n\to\infty} P\left(\bigcup_{k=n}^{\infty} A_k\right)$$ defn B_n

$$\leq \lim_{n\to\infty} \sum_{k=n}^{\infty} P(A_k)$$ Boole's inequality $\sum_{n=1}^{\infty} P(A_n) < \infty$

$$= 0$$

QED (1)

Prf (2):

Suppose $\sum_{n=1}^{\infty} P(A_n) = \infty$ and $\{A_n\}$ independent.

(need the following lemma)

Lemma: $1 - x \leq e^{-x}$ if $x \geq 0$

Prf: define $f(x) := e^{-x} - (1-x) = e^{-x} - 1 + x$

$\therefore f(0) = e^{0} - 1 + 0 = 1 - 1 = 0$

But $f'(x) = -e^{-x} + 1 \geq 0$ for $\forall x \geq 0$

$\therefore f(x) \geq 0$

$\therefore e^{-x} \geq 1 - x$

QED (lemma)

$$P\left(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} A_k\right) = 1 - P\left(\bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} A_k^c\right)$$ DeMorgan's

$$= 1 - P\left(\bigcup_{n=1}^{\infty}\left(\bigcap_{k=n}^{\infty} A_k^c\right)\right)$$ Assoc. $\cup$

$$\geq 1 - \sum_{n=1}^{\infty} P\left(\bigcap_{k=n}^{\infty} A_k^c\right)$$ Boole's inequality

$$\bigcap_{k=n}^{\infty} A_k^c \subset \bigcap_{k=n+1}^{\infty} A_k^c$$

$$= 1 - \sum_{n=1}^{\infty} P\left(\lim_{k\to\infty} A_k^c\right)$$ defn $\lim_{n\to\infty} A_n$

$$= 1 - \sum_{n=1}^{\infty} \lim_{m\to\infty} P\left(\bigcap_{k=n}^{m} A_k^c\right)$$ continuity theorem

$$= 1 - \sum_{n=1}^{\infty} \lim_{m\to\infty} \prod_{k=n}^{m} P(A_k^c)$$ independence

$$= 1 - \sum_{n=1}^{\infty} \lim_{m\to\infty} \prod_{k=n}^{m} (1 - P(A_k))$$ $P(A^c) = 1 - P(A)$

$$\geq 1 - \sum_{n=1}^{\infty} \lim_{m\to\infty} \prod_{k=n}^{m} e^{-P(A_k)}$$ Lemma

$$= 1 - \sum_{n=1}^{\infty} \lim_{m\to\infty} e^{-\sum_{k=n}^{m} P(A_k)}$$ exponential

$$= 1 - \sum_{n=1}^{\infty} e^{-\lim_{m\to\infty} \sum_{k=n}^{m} P(A_k)}$$ exp is continuous

$$= 1 - 0$$ hyp: $\sum_{n=1}^{\infty} P(A_n) = \infty$

$$\therefore P\left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k\right) = 1$$ CAT

QED.

Defn: $\mathcal{B}$ is the tail sigma-algebra of sequence $\{A_1, A_2, \dots\} \subset \mathcal{A}$ on probability space $(\Omega, \mathcal{A}, P)$ if $\mathcal{B} = \bigcap_{n=1}^{\infty} \sigma(A_n, A_{n+1}, \dots)$

CIA.

Thrm: (Kolmogorov's Zero-One Law) Suppose

(1) $A_1, A_2, \ldots$ are independent on $(\Omega, \mathcal{Q}, P)$

(2) $A \in \mathcal{B} = \bigcap_{n=1}^{\infty} \sigma(A_n, A_{n+1}, \ldots)$

"tail event"

Then $P(A) = 0$ or $P(A) = 1$

Ex: - The i.o. event $\underbrace{\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty}}_{} A_k$ is a tail event.

"$\limsup_m A_m$"

$\therefore P\left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k\right) = 0$ or 1 (as Borel-Cantelli)

- Also true for $\underbrace{\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty}}_{} A_k$

"$\liminf_m A_m$"